

A CLASS OF T -STABLE $(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)$ 'S IN G/B

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ABSTRACT. Let G be a connected complex semi-simple group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. We construct a class of smooth T -stable subvarieties inside the flag variety G/B , each of which is an embedding of a product of projective lines.

1. INTRODUCTION

Let G be a connected complex semi-simple group. Let B be a Borel subgroup and consider the natural (left) action of a maximal torus $T \subset B$ on the flag variety G/B . A number of authors have studied T -stable subvarieties in G/B ; see e.g. [2, 3, 4, 5, 6, 8, 9, 10]. In particular, all T -stable curves are known [3]: for each w in the Weyl group $W := N_G(T)/T$ and each root α , there is a unique T -stable curve through the T -fixed points wB and $ws_\alpha B$. Each of these curves is isomorphic to the projective line \mathbb{P}^1 .

In this note, we describe a class of higher dimensional smooth T -stable subvarieties in G/B , generalizing those T -stable curves. More precisely, to each $w \in W$ and each set $\{\alpha_1, \dots, \alpha_d\}$ of pairwise orthogonal roots (in the weaker sense, i.e. the sum of two α_k 's may be a root), we associate a T -stable subvariety in G/B passing through all T -fixed points $w'B$ with w' of the form: w times a product of some of the s_{α_k} . We then show (Theorem 4.1) that such a subvariety is a closed embedding into G/B of a product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ of d projective lines. To the best of my knowledge, these varieties have not yet been described in the literature (except for the curves mentioned above). Also, I do not know whether they exhaust all T -stable subvarieties in G/B that are isomorphic to a product of projective lines.

Although the varieties considered here are reminiscent of Bott-Samelson varieties (see e.g. [1, 7, 11]), we would like to point out some differences: the latter are associated to certain sequences of simple roots (not necessarily orthogonal), whereas our roots need not be simple (but must be orthogonal). Our varieties are direct products of \mathbb{P}^1 's, whereas Bott-Samelson varieties are (generally non-trivial) \mathbb{P}^1 -fibrations over Bott-Samelson varieties of lower dimension. (However, if the roots involved are both simple *and* pairwise orthogonal, then the successive \mathbb{P}^1 -fibrations become trivial and both constructions agree: we obtain the Schubert variety corresponding to the product of the reflections associated to these roots.)

Finally, let us remark that we found the present class of subvarieties of G/B while investigating an approach to quantum analogues of flag varieties [12]. Since they could be of independent interest to algebraic geometers, we decided to describe them in this note, separately from [12].

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2. NOTATION

G	a connected complex semi-simple group
B, B^-	two opposite Borel subgroups of G
T	the maximal torus $B \cap B^-$
U, U^-	the unipotent radicals of B, B^-
W	the Weyl group $N_G(T)/T$
Φ	the root system of G w.r.t. T
Φ^+, Φ^-	the sets of positive and of negative roots w.r.t. B
$<$	the partial order on Φ defined by $\alpha < \beta \iff \beta - \alpha \in \Phi^+$
s_α	the reflection in W associated to a root α
L_α	the copy of $(P)SL(2)$ in G corresponding to a root α
U_α	the root group corresponding to a root α
B_α	the Borel subgroup of L_α containing U_α

3. A CLASS OF SUBVARIETIES IN G/B

Definition 3.1. An *orthocell* in W is a left coset in W of a subgroup generated by pairwise commuting reflections.

To each orthocell $C \subset W$, we will associate a T -stable subvariety $E(C) \subset G/B$. To define it, we will have to make some choices: first, choose $w \in C$, and choose a representative $\dot{w} \in N_G(T)$ of w . By definition, we have $C = w\langle s_\alpha \mid \alpha \in \Omega \rangle$ for some set Ω of positive and pairwise orthogonal roots: next, choose a numbering $\alpha_1, \dots, \alpha_d$ of the elements of Ω that is *nonincreasing*, in the sense that

$$\alpha_k \not\leq \alpha_{k'} \quad \text{for all } k < k'.$$

(This is always possible: choose a maximal element in Ω and call it α_1 , then choose a maximal element among the remaining ones and call it α_2 , etc.) Note that the numbering may be chosen arbitrarily unless Φ has a component of type B_n , C_n , or F_4 .

We then define

$$E(C) := \{\dot{w}g_1 \dots g_d B \mid g_k \in L_{\alpha_k} \forall k\} \subset G/B.$$

In due course, we will show that $E(C)$ only depends on the orthocell C , and not on the choices we have made above (see Remarks 3.5 and 4.2).

Remark 3.2. The set $E(C)$ contains all T -fixed points $w'B$ with $w' \in C$.

Proposition 3.3. *If $\alpha_1, \dots, \alpha_d$ are as above, then the map*

$$L_{\alpha_1} \times \dots \times L_{\alpha_d} \rightarrow G/B : (g_1, \dots, g_d) \mapsto \dot{w}g_1 \dots g_d B$$

factors down to a map

$$j : L_{\alpha_1}/B_{\alpha_1} \times \dots \times L_{\alpha_d}/B_{\alpha_d} \rightarrow G/B.$$

Proof. For all $1 \leq k \leq d$, let $g_k \in L_{\alpha_k}$ and $b_k \in B_{\alpha_k}$. We need to show that $g_1 b_1 \dots g_d b_d B = g_1 \dots g_d B$. By induction over d , we may assume that the left hand side is equal to $g_1 b_1 g_2 \dots g_d B$, so we must show that $g_d^{-1} \dots g_2^{-1} b_1 g_2 \dots g_d \in B$.

We decompose $b_1 = tu$, where $t \in B_{\alpha_1} \cap T$ and $u \in U_{\alpha_1}$. Orthogonality of $\alpha_1, \dots, \alpha_d$ already implies that for each $2 \leq k \leq d$, t commutes with all elements of U_{α_k} and of $U_{-\alpha_k}$. Since $L_{\alpha_k} = \langle U_{\alpha_k}, U_{-\alpha_k} \rangle$ for each k , it follows that $g_d^{-1} \cdots g_2^{-1} t g_2 \cdots g_d = t \in B$.

To study the remaining factor $g_d^{-1} \cdots g_2^{-1} u g_2 \cdots g_d$, we consider the subgroup

$$U_{\alpha_1; \alpha_2, \dots, \alpha_d} := \langle U_{i_1 \alpha_1 + \cdots + i_d \alpha_d} \mid i_1, \dots, i_d \in \mathbf{Z}, i_1 > 0 \rangle$$

(where U_γ denotes the trivial group whenever γ is not a root).

Fix $2 \leq k \leq d$. Let $\beta = i_1 \alpha_1 + \cdots + i_d \alpha_d \in \Phi$ for some $i_1, \dots, i_d \in \mathbf{Z}$, $i_1 > 0$. For all $u_\beta \in U_\beta$ and all $u_{\alpha_k} \in U_{\alpha_k}$, a well known commutation rule (see [13, Proposition 8.2.3]) implies that

$$u_{\alpha_k} u_\beta u_{\alpha_k}^{-1} \in \langle U_{i\beta + j\alpha_k} \mid i > 0, j \geq 0 \rangle \subset U_{\alpha_1; \alpha_2, \dots, \alpha_d}.$$

Similarly, $u_{-\alpha_k} u_\beta u_{-\alpha_k}^{-1} \in U_{\alpha_1; \alpha_2, \dots, \alpha_d}$ for all $u_{-\alpha_k} \in U_{-\alpha_k}$. Using again that $L_{\alpha_k} = \langle U_{\alpha_k}, U_{-\alpha_k} \rangle$, it follows that $g_k^{-1} U_{\alpha_1; \alpha_2, \dots, \alpha_d} g_k = U_{\alpha_1; \alpha_2, \dots, \alpha_d}$.

In particular, $g_d^{-1} \cdots g_2^{-1} u g_2 \cdots g_d \in U_{\alpha_1; \alpha_2, \dots, \alpha_d}$. To complete the proof, it remains to show that $U_{\alpha_1; \alpha_2, \dots, \alpha_d} \subset B$, or, in other words, that $\sum_k i_k \alpha_k$ cannot be a negative root if $i_1 > 0$. Write $\|\alpha\|^2 := (\alpha|\alpha)$ for all α in the root lattice. By orthogonality, we have

$$\|i_1 \alpha_1 + \cdots + i_d \alpha_d\|^2 = i_1^2 \|\alpha_1\|^2 + \cdots + i_d^2 \|\alpha_d\|^2,$$

so there are two cases:

- if α_1 is long (in its component), then $\sum_k i_k \alpha_k$ cannot be a root (except for α_1 itself);
- if α_1 is short, then $\sum_k i_k \alpha_k$ can only be a root if it is of the form $\alpha_1 \pm \alpha_k$ for some $2 \leq k \leq d$. But by assumption, $\alpha_1 \not\prec \alpha_k$, so $\alpha_1 + \alpha_k$ and $\alpha_1 - \alpha_k$ are either positive roots (when $\alpha_1 > \alpha_k$) or nonroots (when α_1, α_k are incomparable).

□

Corollary 3.4. *The set $E(C)$ is a subvariety of the flag variety G/B .*

Proof. Use Proposition 3.3 and the fact that the $L_{\alpha_k}/B_{\alpha_k}$ are complete. □

Remark 3.5. The subvariety $E(C)$ does not depend on the numbering of the elements of Ω (as long as one chooses a nonincreasing one).

Proof. Let β_1, \dots, β_d be another nonincreasing numbering of the elements of Ω . For any incomparable roots γ, γ' , the commutator $(L_\gamma, L_{\gamma'})$ is trivial, so it is enough to show that the sequence $\alpha_1, \dots, \alpha_d$ can be rearranged into β_1, \dots, β_d by successively swapping adjacent pairs of incomparable roots.

We have $\beta_1 = \alpha_k$ for some $1 \leq k \leq d$, and by assumption, this element must be maximal in Ω . Therefore, α_k is incomparable with each of $\alpha_1, \dots, \alpha_{k-1}$, so we may move it past these roots to the beginning of the sequence. Now we have got two sequences with a common first term; discarding it, we may reapply the same procedure inductively. □

4. THE EMBEDDING PROPERTY AND T -STABILITY

We retain all previous notation and assumptions.

Theorem 4.1. *The map j of Proposition 3.3 is an embedding, and its image $E(C)$ is a T -stable subvariety in G/B , isomorphic to a product of d projective lines.*

Proof. First, note that if $E(C)$ is T -stable, then for every $w' \in W$, $E(w'C) = w'E(C)$ is also T -stable; we may therefore restrict the proof to the case where $w = 1$.

Since for each root α , the quotient L_α/B_α is a projective line, the last statement follows from the first one and from Proposition 3.3. Now choose a representative $\dot{s}_\alpha \in N_G(T)$ for the reflection s_α and recall that the Bruhat decomposition $L_\alpha = B_\alpha \cup U_\alpha s_\alpha B_\alpha = B_\alpha \cup s_\alpha U_{-\alpha} B_\alpha$ induces an open covering

$$L_\alpha/B_\alpha = \{uB_\alpha \mid u \in U_{-\alpha}\} \cup \{\dot{s}_\alpha u B_\alpha \mid u \in U_{-\alpha}\}$$

of L_α/B_α by two affine lines.

Taking products, we get an open covering of $\prod_k (L_{\alpha_k}/B_{\alpha_k})$ by 2^d affine sets. More precisely, for each subset $K \subset \{1, \dots, d\}$, we define

$$V_K := \{(\dot{r}_{K,1}u_1B_{\alpha_1}, \dots, \dot{r}_{K,d}u_dB_{\alpha_d}) \mid u_k \in U_{-\alpha_k} \forall k\} \subset \prod_k (L_{\alpha_k}/B_{\alpha_k}),$$

where

$$r_{K,k} := \begin{cases} s_{\alpha_k} & \text{if } k \in K, \\ 1 & \text{if } k \notin K. \end{cases}$$

The set V_\emptyset is dense in $\prod_k (L_{\alpha_k}/B_{\alpha_k})$, so its image

$$j(V_\emptyset) = \{u_1 \dots u_d B \mid u_k \in U_{-\alpha_k} \forall k\}$$

is also dense in $j\left(\prod_k (L_{\alpha_k}/B_{\alpha_k})\right) = E(C)$. Since T normalizes each root group $U_{-\alpha_k}$, it follows that $E(C)$ is T -stable.

It remains to show that j is an embedding. Since the variety $\prod_k (L_{\alpha_k}/B_{\alpha_k})$ is complete, it is enough to show that

- (i) the restriction of j to each open affine set V_K is an embedding, and
- (ii) j is injective.

Condition (i) holds for $K = \emptyset$: indeed, the multiplication map $U_{-\alpha_1} \times \dots \times U_{-\alpha_d} \rightarrow U^-$ is well known to be an embedding (see e.g. [13, §8.2.1]), and so is the canonical map $U^- \rightarrow G/B : u \mapsto uB$ (by the Bruhat decomposition of G).

To show condition (i) for an arbitrary subset $K \subset \{1, \dots, d\}$, we factor the restriction of j to V_K through an isomorphism $V_K \simeq V_\emptyset$, as follows. Consider an element $(\dot{r}_{K,1}u_1B_{\alpha_1}, \dots, \dot{r}_{K,d}u_dB_{\alpha_d}) \in V_K$. Recall that for any two orthogonal roots α, β , s_α normalizes U_β (see e.g. [13, §9.2.1]), so we may rewrite

$$\dot{r}_{K,1}u_1 \dots \dot{r}_{K,d}u_d = (\dot{r}_{K,1} \dots \dot{r}_{K,d})(u'_1 \dots u'_d) = \left(\prod_{k \in K} \dot{s}_{\alpha_k}\right)(u'_1 \dots u'_d)$$

for some $u'_1 \in U_{-\alpha_1}, \dots, u'_d \in U_{-\alpha_d}$. It is clear that the map

$$\sigma_K : V_K \rightarrow V_\emptyset : (\dot{r}_{K,1}u_1B, \dots, \dot{r}_{K,d}u_dB) \mapsto (u'_1B, \dots, u'_dB)$$

is an isomorphism, and by construction, the restriction of j to V_K is equal to the composition $V_K \xrightarrow{\sigma_K} V_\emptyset \xrightarrow{j} G/B \rightarrow G/B$, where the last arrow is (left) multiplication by $\prod_{k \in K} \dot{s}_{\alpha_k}$. This shows condition (i).

Finally, we must check condition (ii), so let $p = (p_1, \dots, p_d)$ and $q = (q_1, \dots, q_d)$ be two points in $\prod_k (L_{\alpha_k}/B_{\alpha_k})$. Let $K \subset \{1, \dots, d\}$ be maximal (w.r.t. set inclusion) such that $p \in V_K$: then for each $k \in K$, $p_k = \dot{s}_{\alpha_k} u_k B_{\alpha_k}$ with $u_k \in U_{-\alpha_k}$, and for each $k \notin K$, we must have $p_k = B_{\alpha_k}$. It follows that

$$j(p) = \left(\prod_{k \in K} \dot{s}_{\alpha_k} u_k \right) B = \left(\prod_{k \in K} u_k^+ \right) \left(\prod_{k \in K} \dot{s}_{\alpha_k} \right) B$$

for some $u_1^+ \in U_{\alpha_1}, \dots, u_d^+ \in U_{\alpha_d}$. Therefore, $j(p)$ lies in the Schubert cell $B \left(\prod_{k \in K} s_{\alpha_k} \right) B/B$. Similarly, if $L \subset \{1, \dots, d\}$ is maximal such that $q \in V_L$, then $j(q) \in B \left(\prod_{k \in L} s_{\alpha_k} \right) B/B$. Now assume $p \neq q$; there are two cases:

- if $K \neq L$, then $j(p) \neq j(q)$ because they lie in different Schubert cells of G/B ;
- if $K = L$, then $j(p) \neq j(q)$ by condition (i).

This shows condition (ii). \square

Remark 4.2. The variety $E(C)$ does not depend on the choice of an element $w \in C$, nor of its representative $\dot{w} \in N_G(T)$.

Proof. The last part follows from T -stability (see Theorem 4.1). For the first part, recall again [13, §9.2.1] that if α, β are orthogonal roots, then $s_\alpha U_\beta s_\alpha^{-1} = U_\beta$; since $L_\beta = \langle U_\beta, U_{-\beta} \rangle$, we then also have $s_\alpha L_\beta s_\alpha^{-1} = L_\beta$. Now use this fact in the definition of $E(C)$. \square

5. TWO EXAMPLES

Example 5.1. $G = \mathrm{SL}(n)$ acts on the set of all (full) flags $F_1 \subset F_2 \subset \cdots \subset F_{n-1}$ (with $\dim F_i = i - 1$) of linear subspaces in the projective space \mathbb{P}^{n-1} . Choose a “hypertetrahedron” in \mathbb{P}^{n-1} , i.e. n linearly independent points $p_1, \dots, p_n \in \mathbb{P}^{n-1}$. Let B be the stabilizer of the flag

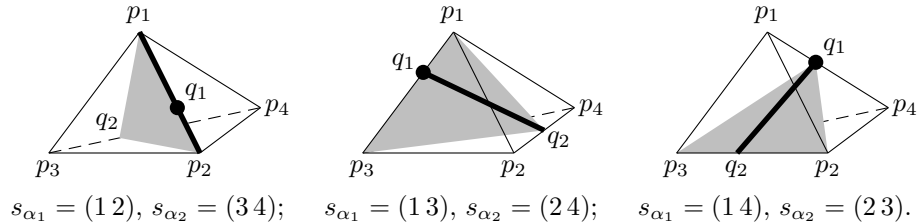
$$\langle p_1 \rangle \subset \langle p_1, p_2 \rangle \subset \cdots \subset \langle p_1, \dots, p_{n-1} \rangle$$

(where $\langle \rangle$ denotes linear span in \mathbb{P}^{n-1}) and let T be the simultaneous stabilizer of the vertices p_1, \dots, p_n . Identifying W with the symmetric group S_n , the T -fixed points in G/B are the flags of the form

$$\langle p_{w(1)} \rangle \subset \langle p_{w(1)}, p_{w(2)} \rangle \subset \cdots \subset \langle p_{w(1)}, \dots, p_{w(n-1)} \rangle$$

for some $w \in W$.

Let $\{\alpha_1, \dots, \alpha_d\}$ be a set of positive and pairwise orthogonal roots; each s_{α_k} is a transposition $(a_k b_k) \in S_n$, and the sequence is automatically nonincreasing. For each k , pick a point q_k on the projective line $\langle p_{a_k}, p_{b_k} \rangle$; to the tuple (q_1, \dots, q_d) , we associate the flag $F(q_1, \dots, q_d)$ whose $(i - 1)$ -dimensional component is obtained from the expression $\langle p_1, \dots, p_i \rangle$ by replacing p_{a_k} by q_k whenever $a_k \leq i < b_k$. For example, when $n = 4$ and $d = 2$, the flag $F(q_1, q_2)$ is of one of the following forms:



Then $E(\langle s_{\alpha_1}, \dots, s_{\alpha_d} \rangle)$ is the set of all $F(q_1, \dots, q_d)$, each q_k varying on the line $\langle p_{a_k}, p_{b_k} \rangle$. If $w \in W \simeq S_n$, then $E(w\langle s_{\alpha_1}, \dots, s_{\alpha_d} \rangle)$ is obtained from the above description by replacing each p_i by $p_{w(i)}$.

(Note that the embedding property of Theorem 4.1 has an easy proof here: indeed, the s_{α_k} pairwise commute, so the sets $\{a_k, b_k\}$ are pairwise disjoint, and therefore the projective lines $\langle p_{a_k}, p_{b_k} \rangle$ on which the q_k vary are pairwise skew.)

Example 5.2. $G = \mathrm{Sp}(4)$ acts on the set of all isotropic flags in \mathbb{P}^3 , i.e. pairs (p, ℓ) with $\ell \subset \mathbb{P}^3$ an isotropic line (w.r.t. the symplectic form defining $\mathrm{Sp}(4)$) and p a point on ℓ . Choose a (skew) “isotropic square” in \mathbb{P}^3 , i.e. four points $p_1, p_2, p_3, p_4 \in \mathbb{P}^3$ such that all lines $\ell_{ij} := \langle p_i, p_j \rangle$ are isotropic except ℓ_{13} and ℓ_{24} . Let B be the stabilizer of the flag (p_1, ℓ_{12}) and T be the simultaneous stabilizer of the vertices p_1, p_2, p_3, p_4 . The T -fixed points in G/B are the flags of the form (p_i, ℓ_{ij}) (with $ij \neq 13$ and $ij \neq 24$).

Let α, β be two positive orthogonal roots. There are two cases.

- If α, β are both short, with $\alpha > \beta$, then $E(\langle s_\alpha, s_\beta \rangle)$ is the set of all isotropic flags (p, ℓ) such that ℓ crosses ℓ_{14} and ℓ_{23} .
- If α, β are both long (hence incomparable), then $E(\langle s_\alpha, s_\beta \rangle)$ is the set of all isotropic flags (p, ℓ) such that p lies on the (nonisotropic) line ℓ_{13} , and ℓ also crosses the (nonisotropic) line ℓ_{24} .

Again, if $w \in W$, then $E(w\langle s_\alpha, s_\beta \rangle)$ is obtained from this description by applying w , viewed as a “symmetry” of the “square” $p_1 p_2 p_3 p_4$.

6. T -ORBITS IN $E(C)$

Let C be an orthocell of rank d and identify $E(C) \simeq \mathbb{P}^1 \times \dots \times \mathbb{P}^1$ via the map j of Proposition 3.3.

Proposition 6.1. *The T -action on $E(C)$ has 3^d orbits, viz. the subsets of the form $A_1 \times \dots \times A_d$, where each $A_k \subset \mathbb{P}^1$ is one of $\{0\}$, $\{\infty\}$, $\mathbb{P}^1 \setminus \{0, \infty\}$.*

Proof. For each $\alpha \in \Phi$, choose an isomorphism $u_\alpha : (\mathbf{C}, +) \rightarrow U_\alpha$ such that $tu_\alpha(z)t^{-1} = u_\alpha(\alpha(t)z)$ for all $t \in T$ and all $z \in \mathbf{C}$ [13, Proposition 8.1.1(i)].

Consider first the case where $C = \langle s_{\alpha_1}, \dots, s_{\alpha_d} \rangle$ (i.e. $w = 1$). The set of all $u_{-\alpha_1}(z_1) \dots u_{-\alpha_d}(z_d)B$, $(z_1, \dots, z_d) \in \mathbf{C}^n$, is an open dense subset of $E(C)$ (cf. the proof of Theorem 4.1), and we have

$$t u_{-\alpha_1}(z_1) \dots u_{-\alpha_d}(z_d)B = u_{-\alpha_1}(\alpha_1(t)^{-1}z_1) \dots u_{-\alpha_d}(\alpha_d(t)^{-1}z_d)B.$$

By orthogonality, the α_k are linearly independent, hence the morphism

$$T \rightarrow \mathbf{C}^* \times \dots \times \mathbf{C}^* : t \mapsto (\alpha_1(t), \dots, \alpha_d(t))$$

is surjective. Therefore, by continuity, this morphism turns the T -action on $E(C)$ into the natural componentwise action of $\mathbf{C}^* \times \dots \times \mathbf{C}^*$ on $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$. The orbit structure is now clear.

The general case $C = w\langle s_{\alpha_1}, \dots, s_{\alpha_d} \rangle$ is obtained similarly, by multiplying $u_{-\alpha_1}(z_1) \dots u_{-\alpha_d}(z_d)B$ by wtw^{-1} . \square

To describe the T -orbit closures, define a *subcell* of an orthocell $C = w\langle s_{\alpha_1}, \dots, s_{\alpha_d} \rangle$ to be an orthocell of the form $w'\langle s_{\alpha'_1}, \dots, s_{\alpha'_e} \rangle$ for some $\{\alpha'_1, \dots, \alpha'_e\} \subset \{\alpha_1, \dots, \alpha_d\}$ and some $w' \in C$.

Corollary 6.2. *The T -orbit closures in $E(C)$ are exactly the $E(C')$ with C' a subcell of C .*

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